

A computation with the Connes-Thom isomorphism

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Abstract

Let $A \in M_n(\mathbb{R})$ be an invertible matrix. Consider the semi-direct product $\mathbb{R}^n \rtimes \mathbb{Z}$ where \mathbb{Z} acts on \mathbb{R}^n by matrix multiplication. Consider a strongly continuous action (α, τ) of $\mathbb{R}^n \rtimes \mathbb{Z}$ on a C^* -algebra B where α is a strongly continuous action of \mathbb{R}^n and τ is an automorphism. The map τ induces a map $\tilde{\tau}$ on $B \rtimes_{\alpha} \mathbb{R}^n$. We show that, at the K -theory level, τ commutes with the Connes-Thom map if $\det(A) > 0$ and anticommutes if $\det(A) < 0$. As an application, we recompute the K -groups of the Cuntz-Li algebra associated to an integer dilation matrix.

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1 Introduction

In [CL10], Cuntz and Li initiated the study of C^* -algebras associated with rings. In [Cun08], Cuntz had earlier studied the C^* -algebra associated to the $ax + b$ group over \mathbb{N} . These C^* -algebras are unital, purely infinite and simple. Thus they are classified by their K -groups. In a series of papers, [CL11], [CL09] and [LL12], the K -groups of these algebras were computed for number fields and for function fields. The main tool used in the K -group computation is the duality result proved in [CL11] and its variations.

Other approaches and possible generalisations were considered in [BE10], [KLQ11] and in [Sun12]. The C^* -algebras studied in [KLQ11] and in [Sun12] were called Cuntz-Li algebras. Following [KLQ11], in [Sun12], the Cuntz-Li algebra associated to a pair $(N \rtimes H, M)$ satisfying certain conditions was studied. Here M is a normal subgroup of N and $N \rtimes H$ is a semidirect product. The main example considered in [Sun12] is the Cuntz-Li algebra, denoted \mathcal{U}_Γ associated to the pair $(\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n)$ where Γ is a subgroup of $GL_n(\mathbb{Q})$ acting by matrix multiplication on \mathbb{Q}^n . In [Sun12], it was proved that \mathcal{U}_Γ is Morita-equivalent to $C(X) \rtimes (\mathbb{R}^n \rtimes \Gamma)$ for some compact Hausdorff space X . This is the analog of the Cuntz-Li duality theorem for the algebra \mathcal{U}_Γ .

A matrix $A \in M_d(\mathbb{Z})$ is called an integer dilation matrix if all its eigenvalues are of absolute value greater than 1. In [EaHR10], a purely infinite simple C^* -algebra associated to an integer dilation matrix was studied and its K -groups were computed. Their computation depends on realising the C^* -algebra as a Cuntz-Pimsner algebra and by a careful examination of the six term sequence coming from its Toeplitz extension. In [MRR], a presentation of this algebra was obtained in terms of generators and relations. For the group $\Gamma := \{(A^t)^r : r \in \mathbb{Z}\} \cong \mathbb{Z}$, denote the Cuntz-Li algebra \mathcal{U}_Γ by \mathcal{U}_{A^t} . The presentation given in [MRR] tells us that the C^* -algebra studied in [EaHR10] is the Cuntz-Li algebra \mathcal{U}_{A^t} .

The purpose of this paper is to understand the K -groups of \mathcal{U}_{A^t} in view of the Cuntz-Li duality theorem. The Cuntz-Li duality theorem in this case says that \mathcal{U}_{A^t} is Morita-equivalent to a crossed product algebra $(C(X) \rtimes \mathbb{R}^d) \rtimes \mathbb{Z}$ for some compact Hausdorff space. We compute the K -groups using the Pimsner-Voiculescu sequence. I believe that this computation will be of independent interest for the following two reasons:

1. The K -groups of \mathcal{U}_{A^t} depends on both d and $\text{sign}(\det(A))$. (Cf. [EaHR10]). The dependence on d is due to the Connes-Thom isomorphism between $K_*(C(X))$ and $K_*(C(X) \rtimes \mathbb{R}^d)$. Also the Connes-Thom map commutes with the action of \mathbb{Z} if $\text{sign}(\det(A)) > 0$ and anticommutes if $\text{sign}(\det(A)) < 0$. This explains the dependence on $\text{sign}(\det(A))$.
2. It is mentioned in the introduction of [CL11] that the duality theorem enables one to use homotopy type arguments which makes it possible to compute the K -groups. We see the same kind of phenomenon here as well. (Cf. Lemma 3.2.)

Let $A \in GL_n(\mathbb{R})$. Consider the semidirect product $\mathbb{R}^n \rtimes \mathbb{Z}$ where \mathbb{Z} acts on \mathbb{R}^n by

matrix multiplication by A . Let B be a C^* -algebra on which $\mathbb{R}^n \rtimes \mathbb{Z}$. The crossed product $B \rtimes (\mathbb{R}^n \rtimes \mathbb{Z})$ is isomorphic to $(B \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$. In section 2 and 3, we write down the Pimsner-Voiculescu sequence for $(B \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$ after identifying the crossed product $B \rtimes \mathbb{R}^n$ with B upto KK -equivalence. We show that the Connes-Thom isomorphism commutes with the action of \mathbb{Z} if $\det(A) > 0$ and anticommutes if $\det(A) < 0$. In section 4 and 5, the K -groups of \mathcal{U}_{A^t} were (re)computed.

2 Preliminaries

We use this section to fix notations and recall a few preliminaries.

Let $A \in M_n(\mathbb{R})$ be such that $\det(A) \neq 0$. We think of elements of \mathbb{R}^n as column vectors. Thus the matrix A induces an action of \mathbb{Z} on \mathbb{R}^n by left multiplication. The generator $1 \in \mathbb{Z}$ acts on \mathbb{R}^n by $1.v = Av$ for $v \in \mathbb{R}^n$. Consider the semidirect product $\mathbb{R}^n \rtimes \mathbb{Z}$.

Let B be a C^* -algebra. A strongly continuous action of $\mathbb{R}^n \rtimes \mathbb{Z}$ on B is equivalent to providing a pair (α, τ) where α is a strongly continuous action of \mathbb{R}^n on B and τ is an automorphism of B such that

$$\tau\alpha_\xi = \alpha_{A\xi}\tau$$

for every $\xi \in \mathbb{R}^n$. If (α, τ) is such a pair, we write $\alpha \rtimes \tau$ for the action of $\mathbb{R}^n \rtimes \mathbb{Z}$. Also the automorphism τ induces an action, denoted $\tilde{\tau}$, on the crossed product $B \rtimes_\alpha \mathbb{R}^n$ given by

$$\begin{aligned} \tilde{\tau}(b) &:= \tau(b) \text{ if } b \in B \\ \tilde{\tau}(U_\xi) &:= U_{A\xi} \text{ for } \xi \in \mathbb{R}^n \end{aligned}$$

where U_ξ denotes the canonical unitary in $\mathcal{M}(B \rtimes_\alpha \mathbb{R}^n)$. Moreover the crossed product $B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})$ is isomorphic to $(B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\tilde{\tau}} \mathbb{Z}$.

The Pimsner-Voiculescu sequence gives the following six-term exact sequence.

$$\begin{array}{ccccc} K_0(B \rtimes_\alpha \mathbb{R}^n) & \xrightarrow{1-\tilde{\tau}_*} & K_0(B \rtimes_\alpha \mathbb{R}^n) & \longrightarrow & K_0(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) & \longleftarrow & K_1(B \rtimes_\alpha \mathbb{R}^n) & \xleftarrow{1-\tilde{\tau}_*} & K_1(B \rtimes_\alpha \mathbb{R}^n) \end{array}$$

But by the Connes-Thom isomorphism, we can replace $K_i(B \rtimes_\alpha \mathbb{R}^n)$ by $K_{i+n}(B)$. Let $C_{n,i} : K_i(B) \rightarrow K_{i+n}(B \rtimes_\alpha \mathbb{R}^n)$ be the Connes-Thom map. Now we can state our main theorem.

Theorem 2.1 *For $i = 1, 2$, $C_{n,i}\tau_* = \epsilon \tilde{\tau}_* C_{n,i}$ where ϵ is given by*

$$\epsilon := \begin{cases} 1 & \text{if } \det(A) > 0 \\ -1 & \text{if } \det(A) < 0 \end{cases}$$

The following is an immediate corollary to Theorem 2.1.

Corollary 2.2 *Let (α, τ) be a strongly continuous action of $\mathbb{R}^n \rtimes \mathbb{Z}$ on a C^* -algebra B . Then there exists a six term exact sequence*

$$\begin{array}{ccccc} K_n(B) & \xrightarrow{1-\epsilon\tau_*} & K_n(B) & \longrightarrow & K_0(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) & \longleftarrow & K_{n+1}(B) & \xleftarrow{1-\epsilon\tau_*} & K_{n+1}(B) \end{array}$$

where $\epsilon = \text{sign}(\det(A))$.

3 Proof of Theorem 2.1

We use KK -theory to prove this. All our algebras are ungraded. We denote the interior Kasparov product

$$KK^{(i)}(A, B) \times KK^{(j)}(B, C) \rightarrow KK^{(i+j)}(A, C)$$

by \sharp and the external Kasparov product

$$KK^{(i)}(A_1, A_2) \times KK^{(j)}(B_1, B_2) \rightarrow KK^{(i+j)}(A_1 \otimes A_2, B_1 \otimes B_2)$$

by $\hat{\otimes}$. We will also identify $K_i(B)$ with $KK^{(i)}(\mathbb{C}, B)$. Also if $\phi : B_1 \rightarrow B_2$ is a C^* -algebra homomorphism then we denote the KK -element $(B_2, \phi, 0)$ in $KK^{(0)}(B_1, B_2)$ by $[\phi]$.

Under this identification, the Connes-Thom isomorphism is given by $C_n(x) = x \sharp t_\alpha$ where $t_\alpha \in KK^n(B, B \rtimes_\alpha \mathbb{R}^n)$ is the Thom element.

Now it is immediate that Theorem 2.1 is equivalent to the following theorem.

Theorem 3.1 *One has $[\tau] \sharp t_\alpha = \epsilon t_\alpha \sharp [\tilde{\tau}]$ where $\epsilon = \text{sign}(\det(A))$.*

A bit of notation. If $X \in GL_n(\mathbb{R})$ then X induces an automorphism ϕ_X on $C_0(\mathbb{R}^n)$ given by $(\phi_X f)(v) := f(Xv)$. Let $b_n \in K_n(C_0(\mathbb{R}^n))$ be the Bott element. We denote the image $\phi_{X*}(b_n) \in K_n(C_0(\mathbb{R}^n))$ simply by $X_*(b_n)$.

First let us dispose of the case when the action of \mathbb{R}^n is trivial. For the trivial action the crossed product $B \rtimes_{\alpha} \mathbb{R}^n$ is isomorphic to $B \otimes C_0(\mathbb{R}^n)$ and $t_{\text{trivial}} = 1_B \widehat{\otimes} b_n$.

Lemma 3.2 *If the action of \mathbb{R}^n is trivial, then $[\tau] \sharp t_{\text{trivial}} = \epsilon t_{\text{trivial}} \sharp [\widetilde{\tau}]$ where $\epsilon = \text{sign}(\det(A))$.*

Proof. Note that

$$\begin{aligned} [\tau] \sharp t_{\text{trivial}} &= [\tau] \widehat{\otimes} b_n \\ t_{\text{trivial}} \sharp [\widetilde{\tau}] &= [\tau] \widehat{\otimes} A_*^t(b_n) \end{aligned}$$

Thus we only need to prove that $A_*^t(b_n) = \epsilon b_n$ where $\epsilon = \text{sign}(\det(A))$.

If $\det(A) > 0$, then A^t is homotopic to identity in $GL_n(\mathbb{R})$. Hence $A_*^t(b_n) = b_n$.

If $\det(A) < 0$, then A^t is homotopic to $\begin{pmatrix} -1 & 0 \\ 0 & Id_{n-1} \end{pmatrix}$ in $GL_n(\mathbb{R})$. The Bott element $b_n = b_1 \widehat{\otimes} b_1 \widehat{\otimes} \cdots \widehat{\otimes} b_1$, it follows that the matrix $\begin{pmatrix} -1 & 0 \\ 0 & Id_{n-1} \end{pmatrix}$ sends b_n to $-b_n$. As a consequence, we have $A_*^t(b_n) = -b_n$ if $\det(A) < 0$. This completes the proof. \square

Now by a homotopy argument, the argument that is used in Theorem 2 of [FS81], we reduce Theorem 3.1 to Lemma 3.2.

For $s \in [0, 1]$, let α^s be the action of \mathbb{R}^n on B defined by $\alpha_{\xi}^s(b) := \alpha_{s\xi}(b)$. Note that $\alpha^1 = \alpha$ and α^0 gives the trivial action. Observe that $\tau \alpha_{\xi}^s = \alpha_{A\xi}^s \tau$. For $s \in [0, 1]$, denote the automorphism τ by τ^s and the automorphism induced by τ on $B \rtimes_{\alpha^s} \mathbb{R}^n$ by $\widetilde{\tau}^s$.

Let $IB := C[0, 1] \otimes B$. Consider the action $\underline{\alpha}$ of \mathbb{R}^n and the automorphism $\underline{\tau}$ on IB defined by

$$\begin{aligned} \underline{\alpha}_{\xi}(f)(s) &:= \alpha_{\xi}^s(f(s)) \\ \underline{\tau}(f)(s) &= \tau(f(s)) \end{aligned}$$

Observe that for $\xi \in \mathbb{R}^n$, $\underline{\tau} \underline{\alpha}_{\xi} = \underline{\alpha}_{A\xi} \underline{\tau}$. The automorphism $\underline{\tau}$ induces an automorphism on $IB \rtimes_{\underline{\alpha}} \mathbb{R}^n$ and we denote it by $\widetilde{\underline{\tau}}$.

For $s \in [0, 1]$, let $\epsilon_s : IB \rightarrow B$ be the evaluation map. Then $\epsilon_s : (IB, \underline{\alpha}) \rightarrow (B, \alpha^s)$ is equivariant. We denote the induced map from $IB \rtimes_{\underline{\alpha}} \mathbb{R}^n$ to $B \rtimes_{\alpha^s} \mathbb{R}^n$ by $\widehat{\epsilon}_s$.

Also for $s \in [0, 1]$, $\widehat{\epsilon}_s \circ \widetilde{\underline{\tau}} = \widetilde{\tau}^s \circ \widehat{\epsilon}_s$.

Lemma 3.3 *For $s \in [0, 1]$, the element $[\widehat{\epsilon}_s] \in KK^{(0)}(IB \rtimes_{\underline{\alpha}} \mathbb{R}^n, B \rtimes_{\alpha^s} \mathbb{R}^n)$ is a KK -equivalence.*

Proof. Observe that

$$t_{\underline{\alpha}} \# [\widehat{\epsilon}_s] = [\epsilon_s] \# [t_{\alpha^s}].$$

Since $[\epsilon_s] \in KK^{(0)}(IB, B)$ and the Thom elements are KK -equivalences, it follows that $[\widehat{\epsilon}_s]$ is a KK -equivalence. This completes the proof. \square

Proposition 3.4 *The following are equivalent. Recall that $\epsilon = \text{sign}(\det(A))$.*

1. *For every $s \in [0, 1]$, $[\tau^s] \# t_{\alpha^s} = \epsilon t_{\alpha^s} \# [\widetilde{\tau}^s]$.*
2. *There exists $s \in [0, 1]$ such that $[\tau^s] \# t_{\alpha^s} = \epsilon t_{\alpha^s} \# [\widetilde{\tau}^s]$.*
3. *The Kasparov product $[\mathcal{T}] \# [t_{\underline{\alpha}}] = \epsilon t_{\underline{\alpha}} \# [\widetilde{\mathcal{T}}]$.*

Proof. Let $s \in [0, 1]$ be given. Observe the following.

$$\begin{aligned} [\mathcal{T}] \# t_{\underline{\alpha}} &= \epsilon t_{\underline{\alpha}} \# [\widetilde{\mathcal{T}}] \\ &\Leftrightarrow [\mathcal{T}] \# t_{\underline{\alpha}} \# [\widehat{\epsilon}_s] = \epsilon t_{\underline{\alpha}} \# [\widetilde{\mathcal{T}}] \# [\widehat{\epsilon}_s] \text{ (Since } [\widehat{\epsilon}_s] \text{ is a } KK\text{-equivalence.)} \\ &\Leftrightarrow [\mathcal{T}] \# [\epsilon_s] \# t_{\alpha^s} = \epsilon t_{\underline{\alpha}} \# [\widehat{\epsilon}_s \circ \mathcal{T}] \\ &\Leftrightarrow [\epsilon_s \circ \mathcal{T}] \# t_{\alpha^s} = \epsilon t_{\underline{\alpha}} \# [\widetilde{\tau}^s \circ \widehat{\epsilon}_s] \\ &\Leftrightarrow [\tau^s \circ \epsilon_s] \# t_{\alpha^s} = \epsilon t_{\underline{\alpha}} \# [\widehat{\epsilon}_s] \# [\widetilde{\tau}^s] \\ &\Leftrightarrow [\epsilon_s] \# [\tau^s] \# t_{\alpha^s} = \epsilon [\epsilon_s] \# t_{\alpha^s} \# [\widetilde{\tau}^s] \\ &\Leftrightarrow [\tau^s] \# [t_{\alpha^s}] = \epsilon t_{\alpha^s} \# [\widetilde{\tau}^s] \text{ (Since } [\epsilon_s] \text{ is a } KK\text{-equivalence.)} \end{aligned}$$

The proof is now complete. \square .

Now Theorem 3.1 follows from Proposition 3.4 and Lemma 3.2.

4 The Cuntz-Li algebra associated to an integer dilation matrix

As an application of Corollary 2.2, we recompute the K -theory of the C^* -algebra associated to an integer dilation matrix, studied in [EaHR10]. Let us recall the C^* -algebra considered in [EaHR10].

Let $A \in M_d(\mathbb{Z})$ be an integer dilation matrix i.e. all the eigen values of A are of absolute value greater than 1. The matrix A acts on \mathbb{R}^d by matrix multiplication and leaves \mathbb{Z}^d invariant. Denote the resulting endomorphism on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ by σ_A . The map σ_A is a surjective and has finite fibres. Denote the map $C(\mathbb{T}^d) \ni f \rightarrow f \circ \sigma_A \in C(\mathbb{T}^d)$ by α_A . Consider the transfer operator $L : C(\mathbb{T}^d) \rightarrow C(\mathbb{T}^d)$ defined by

$$L(f)(x) := \frac{1}{|\sigma_A^{-1}(x)|} \sum_{\sigma_A(y)=x} f(y)$$

Then L satisfies the condition $L(\alpha_A(f)g) = fL(g)$ for $f, g \in C(\mathbb{T}^d)$. In [EaHR10], the Exel Crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ was viewed as a Cuntz-Pimsner algebra $\mathcal{O}(M_L)$ of a suitable Hilbert $C(\mathbb{T}^d)$ bimodule M_L .

By a careful examination of the six term sequence (and the maps involved) associated to the exact sequence $0 \rightarrow \text{Ker}(Q) \rightarrow \mathcal{T}(M_L) \rightarrow \mathcal{O}(M_L)$, the K -groups of $C(\mathbb{T}^d) \rtimes_{\alpha_A} \mathbb{N}$ were computed in [EaHR10].

For our purposes, the following description of $\mathcal{O}(M_L)$ in terms of generators and relations is more relevant. Let us recall the following proposition from [MRR] (Proposition 3.3, Page 6).

Proposition 4.1 ([MRR]) *The Exel's crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ is the universal C^* -algebra generated by an isometry v and unitaries $\{u_m : m \in \mathbb{Z}^d\}$ satisfying the following relations.*

$$\begin{aligned} u_m u_n &= u_{m+n} \\ v u_m &= u_{A^t m} v \\ \sum_{m \in \Sigma} u_m v v^* u_m^{-1} &= 1 \end{aligned}$$

Here Σ denotes a set of distinct coset representatives of the group $\mathbb{Z}^d/A^t \mathbb{Z}^d$.

Remark 4.2 *The above relations are called conditions (E1) and (E3) in [MRR]. Condition (E2) in [MRR] is implied by (E1) and (E3).*

For if $m \notin A^t \mathbb{Z}^d$, the projections vv^* and $u_m v v^* u_m^{-1}$ are orthogonal by (E3). Hence $v^* u_m v = 0$ if $m \notin A^t \mathbb{Z}^d$. If $m \in A^t \mathbb{Z}^d$, then using $v u_m = u_{A^t m} v$, one obtains $v^* u_m v = u_{(A^t)^{-1}m}$. Thus (E1) and (E3) implies (E2).

The following setup was initially considered in [KLQ11]. Consider a semi-direct product $N \rtimes H$ and let M be a normal subgroup. Let $P := \{a \in H : aMa^{-1} \subset M\}$. Then P is a semigroup containing the identity e . For $a \in P$, let $M_a = aMa^{-1}$. Assume that the following holds.

(C1) The group $H = PP^{-1} = P^{-1}P$.

(C2) For every $a \in P$, the subgroup aMa^{-1} is of finite index in M .

(C3) The intersection $\bigcap_{a \in P} aMa^{-1} = \{e\}$ where e denotes the identity element of G .

Definition 4.3 *The Cuntz-Li algebra associated to the pair $(N \rtimes H, M)$ is the universal C^* -algebra generated by a set of isometries $\{s_a : a \in P\}$ and a set of unitaries $\{u(m) : m \in M\}$ satisfying the following relations.*

$$\begin{aligned} s_a s_b &= s_{ab} \\ u(m)u(n) &= u(mn) \\ s_a u(m) &= u(ama^{-1})s_a \\ \sum_{k \in M/M_a} u(k)e_a u(k)^{-1} &= 1 \end{aligned}$$

where e_a denotes the final projection of s_a . We denote the Cuntz-Li algebra associated to the pair $(N \rtimes H, M)$ by $U[N \rtimes H, M]$.

Let $A \in M_d(\mathbb{Z})$ be a dilation matrix. Then A acts on \mathbb{Q}^d by left multiplication. Consider the semidirect product $\mathbb{Q}^d \rtimes \mathbb{Z}$ and the normal subgroup \mathbb{Z}^d of \mathbb{Q}^d . For this pair $(\mathbb{Q}^d \rtimes \mathbb{Z}, \mathbb{Z}^d)$, $P = \{A^r : r \geq 0\} \cong \mathbb{N}$. Moreover conditions (C1) – (C3) are satisfied. (See Example 2.6, Page 3 in [Sun12].) Let us denote the Cuntz-Li algebra $U[\mathbb{Q}^d \rtimes \mathbb{Z}, \mathbb{Z}^d]$ simply by \mathcal{U}_A .

By using the presentation (Cf. Prop. 4.1) of the Exel's Crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ given in terms of isometries and unitaries, it is easy to verify that $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ is isomorphic to \mathcal{U}_A .

Let us recall the Cuntz-Li duality result proved in [Sun12]. The proof is really a step by step adaptation of the arguments used in [CL11].

Let $N_A := \bigcup_{r=0}^{\infty} A^{-r} \mathbb{Z}^d$. Then N_A is a subgroup of \mathbb{R}^d . Let \mathbb{Z} act on \mathbb{R}^d by left multiplication by A^t and consider the semidirect product $\mathbb{R}^d \rtimes \mathbb{Z}$. The semidirect product

$\mathbb{R}^d \rtimes \mathbb{Z}$ acts on the C^* -algebra $C^*(N_A)$. The action α of \mathbb{R}^d and the automorphism τ , corresponding to the action of \mathbb{Z} , are given by

$$\begin{aligned}\alpha_\xi(\delta_v) &:= e^{-2\pi i \langle \xi, v \rangle} \delta_v \text{ for } \xi \in \mathbb{R}^d \\ \tau(\delta_v) &:= \delta_{A^{-1}v}\end{aligned}$$

where $\{\delta_v : v \in N_A\}$ denotes the canonical unitaries in $C^*(N_A)$ and \langle, \rangle denotes the usual inner product on \mathbb{R}^d . Note that $\tau\alpha_\xi = \alpha_{A^t\xi}\tau$ for $\xi \in \mathbb{R}^d$.

The following proposition was proved in [Sun12]. (Cf. Theorem 8.2 and Proposition 8.6 in [Sun12])

Proposition 4.4 *The C^* -algebra \mathcal{U}_{A^t} is Morita-equivalent to $C^*(N_A) \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^d \rtimes_{A^t} \mathbb{Z})$.*

Now using the Morita-equivalence in Proposition 4.4 and using the version of Pimsner-Voiculescu exact sequence established in Corollary 2.2, the K -groups of \mathcal{U}_{A^t} can be computed.

5 K -groups of the Cuntz-Li algebra \mathcal{U}_{A^t}

Recall that $N_A := \bigcup_{r=0}^\infty A^{-r}\mathbb{Z}^d$. Set $N_A^{(r)} := A^{-r}\mathbb{Z}^d$. Then $\{N_A^{(r)}\}_{r=0}^\infty$ forms an increasing sequence of subgroups, each isomorphic to \mathbb{Z}^d and $N_A = \bigcup_{r=0}^\infty N_A^{(r)}$. Thus $C^*(N_A)$ is the inductive limit of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$. Thus $K_*(C^*(N_A))$ can be computed as the inductive limit of the K -groups of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$.

Let us first recall the K -theory of $C(\mathbb{T}^d) \cong C^*(\mathbb{Z}^d)$. It is well known and can be proved by the Kunneth formula that as a \mathbb{Z}_2 graded ring, $K_*(C^*(\mathbb{Z}^d))$ is isomorphic to the exterior algebra $\Lambda^*(\mathbb{Z}^d)$. The map $\mathbb{Z}^d \ni e_i \rightarrow \delta_{e_i} \in K_1(C^*(\mathbb{Z}^d))$ extends to a graded ring isomorphism from $\Lambda^*(\mathbb{Z}^d)$ to $K_*(C^*(\mathbb{Z}^d))$.

Remark 5.1 *The isomorphism $K_*(C(\mathbb{T}^d)) \cong \Lambda^*\mathbb{Z}^d$ was also used in [EaHR10].*

Let us now fix some notations. For $0 \leq n \leq d$, let A_n be the map on $\Lambda^n(\mathbb{Z}^d)$ induced by A . Thus $A_0 = 1$, $A_1 = A$ and $A_d = \det(A)$. For a subset I of $\{1, 2, \dots, d\}$, of cardinality n , (arranged in increasing order), $I = \{i_1 < i_2 < \dots, < i_n\}$, let $e_I := e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n}$. Then $\{e_I : |I| = n\}$ is a basis for $\Lambda^n(\mathbb{Z}^d)$. For subsets J, K of $\{1, 2, \dots, d\}$ of size n , let $A_{J,K}$ be the submatrix of A obtained by considering the rows coming from J and

columns coming from K . With respect to the basis $\{e_I : |I| = n\}$, the $(J, K)^{th}$ entry of the matrix corresponding to A_n is $\det(A_{J,K})$.

Note that for $n \geq 1$, A_n is again an integer dilation matrix. For if we upper triangularise A , then w.r.t the basis $\{e_I : |I| = n\}$, arranged in lexicographic order, A_n is upper triangular and the eigen values of A_n are product of eigen values of A . Thus the eigen values of A_n are of absolute value greater than 1. This fact was used in [EaHR10]. (Cf. Proposition 4.6. in [EaHR10].)

Let $n \in \{0, 1, 2, \dots, n\}$. Consider $\Lambda^n(\mathbb{Z}^d)$ as a subgroup of $\Lambda^n(\mathbb{Q}^d)$. Then A_n is invertible on $\Lambda^n(\mathbb{Q}^d)$. Set $\Gamma_n := \bigcup_{r=0}^{\infty} A_n^{-r}(\Lambda^n(\mathbb{Z}^d))$.

Proposition 5.2 *The K -groups of the C^* -algebra $C^*(N_A)$ are given by*

$$K_0(C^*(N_A)) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \Gamma_n$$

$$K_1(C^*(N_A)) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \Gamma_n$$

Proof. Since $C^*(N_A)$ is the inductive limit of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$, it follows that $K_*(C^*(N_A))$ is the inductive limit of $K_*(C^*(N_A^{(r)})) \cong K_*(C(\mathbb{T}^d))$. Identify $K_*(C^*(N_A^{(r)}))$ with $\Lambda^*(\mathbb{Z}^d)$ via the map $\delta_{A^{-r}(e_i)} \rightarrow e_i$. With this identification the inclusion map $C^*(N_A^{(r)}) \rightarrow C^*(N_A^{(r+1)})$ induces the map $\bigoplus_{1 \leq n \leq d} A_n$ at the K -theory level. (Reason: If we write e_j as a linear combination of $\{A^{-1}e_i\}$ the matrix involved is just A .)

Thus we are left to show that the inductive limit of $(\bigoplus_n \Lambda^n \mathbb{Z}^d, \bigoplus_n A_n)_{r=0}^{\infty}$ is $\bigoplus_n \Gamma_n$. Again it is enough to show that the inductive limit of $(\Lambda^n \mathbb{Z}^d, A_n)_{r=0}^{\infty}$ is isomorphic to Γ_n . Let $H_r = \Lambda^n \mathbb{Z}^d$. If $v \in \Gamma_n$, write v as $v = A_n^{-r} w$ with $w \in \Lambda^n(\mathbb{Z}^d)$. The map $\Gamma_n \ni v \rightarrow w \in H_r$ is an isomorphism between Γ_n and $\lim_{r \rightarrow \infty} (\Lambda^n(\mathbb{Z}^d), A_n)$. This completes the proof. \square

Now let us calculate the automorphism τ on $C^*(N_A)$. Recall that τ on the generating unitaries is given by $\tau(\delta_v) = \delta_{A^{-1}v}$. Thus, it is immediate and not difficult to see that τ induces the map $\bigoplus_n A_n^{-1}$ on $\bigoplus_n \Gamma_n$ when one identifies $K_*(C^*(N_A))$ with $\bigoplus_n \Gamma_n$ (together with their \mathbb{Z}_2 grading).

We need one more lemma.

Lemma 5.3 *Let $1 \leq n \leq d$. The natural map*

$$\frac{\Lambda^n(\mathbb{Z}^d)}{(1 - \epsilon A_n)(\Lambda^n(\mathbb{Z}^d))} \rightarrow \frac{\Gamma_n}{(1 - \epsilon A_n)\Gamma_n}$$

is an isomorphism for $\epsilon \in \{1, -1\}$.

Proof. Let us denote $\Lambda^n(\mathbb{Z}^d)$ by V_n . We will give a proof only for $\epsilon = 1$. The case $\epsilon = -1$ is similar and we leave its proof to the reader.

Observe that for $1 - A_n^r = (1 - A_n)(1 + A_n + A_n^2 + \cdots + A_n^{r-1})$ for $r \geq 0$. Thus for every $r \geq 0$, there exists a polynomial $p_r(x)$ with integer co-efficients such that $p_r(A_n)(1 - A_n) + A_n^r = 1$.

Surjectivity: Let $v \in \Gamma_n$ be given. By definition, there exists $r \geq 0$ and $w \in V_n$ such that $v = A_n^{-r}w$. Hence

$$\begin{aligned} v &= A_n^{-r}(A_n^r + p_r(A_n)(1 - A_n))w \\ &= w + (1 - A_n)p_r(A_n)A_n^{-r}w \end{aligned}$$

Hence $v \equiv w \pmod{(1 - A_n)\Gamma_n}$. This proves the surjectivity of the given map.

Injectivity: Let $v \in V_n$ be such that $v \equiv 0 \pmod{(1 - A_n)\Gamma_n}$. This implies that there exists $r \geq 0$ and $w \in \Lambda^n(\mathbb{Z}^d)$ such that $v = (1 - A_n)A_n^{-r}w$. Hence $(1 - A_n)w = A_n^r v$. Now observe that

$$\begin{aligned} w &= (p_r(A_n)(1 - A_n) + A_n^r)w \\ &= p_r(A_n)A_n^r v + A_n^r w \\ &= A_n^r(p_r(A_n)v + w) \end{aligned}$$

Hence $A_n^{-r}w = p_r(A_n)v + w \in V_n$. Hence $v \equiv 0 \pmod{(1 - A_n)V_n}$. This proves the injectivity part. This completes the proof. \square

We denote both the abelian groups $\frac{\Lambda^n(\mathbb{Z}^d)}{(1 - \epsilon A_n)(\Lambda^n(\mathbb{Z}^d))}$ and $\frac{\Gamma_n}{(1 - \epsilon A_n)\Gamma_n}$ by $\text{coker}(1 - \epsilon A_n)$.

Theorem 5.4 *Let $A \in M_d(\mathbb{Z})$ be an integer dilation matrix. The K -groups of the Cuntz-Li algebra \mathcal{U}_{A^t} are as follows.*

1. *If d is even and $\det(A) > 0$, then*

$$\begin{aligned} K_0(\mathcal{U}_{A^t}) &\cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \text{coker}(1 - A_n), \text{ and} \\ K_1(\mathcal{U}_{A^t}) &\cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \text{coker}(1 - A_n) \oplus \mathbb{Z}. \end{aligned}$$

2. If d is even and $\det(A) < 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \text{coker}(1 + A_n), \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \text{coker}(1 + A_n).$$

3. If d is odd and $\det(A) > 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \text{coker}(1 - A_n) \oplus \mathbb{Z}, \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \text{coker}(1 - A_n).$$

4. If d is odd and $\det(A) < 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \text{coker}(1 + A_n), \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \text{coker}(1 + A_n).$$

Proof. Our main tool is Corollary 2.2 and the Morita equivalence between \mathcal{U}_{A^t} and $C^*(N_A) \rtimes (\mathbb{R}^n \rtimes \mathbb{Z})$.

If d is even and $\det(A) > 0$ then by Corollary 2.2, one has the following six term exact sequence.

$$\begin{array}{ccccc} K_0(C^*(N_A)) & \xrightarrow{1-\tau_*} & K_0(C^*(N_A)) & \longrightarrow & K_0(\mathcal{U}_{A^t}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{U}_{A^t}) & \longleftarrow & K_1(C^*(N_A)) & \xleftarrow{1-\tau_*} & K_1(C^*(N_A)) \end{array}$$

Now by Prop. 5.2, the above six term sequence becomes

$$\begin{array}{ccccc} \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \Gamma_n & \xrightarrow{1-\oplus_n A_n^{-1}} & \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \Gamma_n & \longrightarrow & K_0(\mathcal{U}_{A^t}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{U}_{A^t}) & \longleftarrow & \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \Gamma_n & \xleftarrow{1-\oplus_n A_n^{-1}} & \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \Gamma_n \end{array}$$

Now for $n \geq 1$, A_n is a dilation matrix and thus $\ker(1 - A_n^{-1}) = 0$ if $n \geq 1$. Hence we conclude from the above six term sequence that

$$K_0(\mathcal{U}_{A^t}) \equiv \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n^{-1}).$$

Since A_n is invertible and $(1 - A_n) = -A_n(1 - A_n^{-1})$, it follows that $\operatorname{coker}(1 - A_n) \equiv \operatorname{coker}(1 - A_n^{-1})$. Thus

$$K_0(\mathcal{U}_{A^t}) \equiv \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n^{-1}) \equiv \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n).$$

Now $A_0 = 1$ and hence $\bigoplus_{n \text{ even}} \ker(1 - A_n^{-1}) = \mathbb{Z}$. Again the six term sequence gives the following short exact sequence.

$$0 \longrightarrow \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n^{-1}) \longrightarrow K_1(\mathcal{U}_{A^t}) \longrightarrow \mathbb{Z}$$

Since \mathbb{Z} is free, it follows that

$$K_1(\mathcal{U}_{A^t}) \equiv \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n^{-1}) \oplus \mathbb{Z} \equiv \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - A_n) \oplus \mathbb{Z}.$$

If d is even and $\det(A) < 0$ then by Corollary 2.2 and 5.2, we get the following six term sequence.

$$\begin{array}{ccccc} \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \Gamma_n & \xrightarrow{1 + \bigoplus_n A_n^{-1}} & \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \Gamma_n & \longrightarrow & K_0(\mathcal{U}_{A^t}) \\ & & & & \downarrow \\ K_1(\mathcal{U}_{A^t}) & \longleftarrow & \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \Gamma_n & \xleftarrow{1 + \bigoplus_n A_n^{-1}} & \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \Gamma_n \end{array}$$

Again for $n \geq 1$, A_n is a dilation matrix and for $n = 0$, $A_0 = 1$. Hence for $0 \leq n \leq d$, $\ker(1 + A_n^{-1}) = 0$. Thus the above six term sequence implies that

$$\begin{aligned} K_0(\mathcal{U}_{A^t}) &\equiv \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 + A_n^{-1}) \equiv \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 + A_n) \\ K_1(\mathcal{U}_{A^t}) &\equiv \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 + A_n^{-1}) \equiv \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 + A_n) \end{aligned}$$

The case when d is odd is similar (again an application of Corollary 2.2) and we leave the details to the reader. This completes the proof. \square

The rest of this section is devoted to reconciling Theorem 5.5 with the result obtained in [EaHR10]. More precisely with Theorem 4.9 of [EaHR10]. Let us recall the notations as in [EaHR10].

For a subset $K = \{k_1 < k_2 < \cdots < k_n\}$ of $\{1, 2, \dots, d\}$, denote the complement arranged in increasing order by K' and let $K' = \{k_{n+1} < k_{n+2} < \cdots < k_d\}$. Denote the permutation $i \rightarrow k_i$ by τ_K . For a permutation σ , $\text{sign}(\sigma)$ is 1 if σ is even and -1 if $\text{sign}(\sigma)$ is odd. Also recall that if K and J are subsets of size n , then $A_{K,J}$ is the matrix obtained from A by considering the rows from K and columns from J .

For $0 \leq n \leq d$, let \tilde{B}_n be the $\binom{d}{n} \times \binom{d}{n}$ matrix defined as follows. (We index the columns and rows by subsets of $\{1, 2, \dots, d\}$ of size n .) The $(K, L)^{th}$ entry of \tilde{B}_n is $\text{sign}(\tau_K \tau_L) \det(A_{K', L'})$.

The matrices B_n as defined in [EaHR10] (Prop 4.6.) are then given by $B_n = \text{sign}(\det(A)) \tilde{B}_n$. Denote the matrix whose $(K, L)^{th}$ entry is $\det(A_{K', L'})$ by C_n . By convention, $\tilde{B}_d = 1 = C_d$. Note that \tilde{B}_n and C_n are conjugate over \mathbb{Z} . For the matrix $\text{diag}(\text{sign}(\tau_K))$ conjugates \tilde{B}_n to C_n .

Let $U_n : \Lambda^n(\mathbb{Z}^d) \rightarrow \Lambda^{d-n}(\mathbb{Z}^d)$ be defined by $U_n e_I := e_{I'}$. Then U_n is invertible and $U_n C_n U_n^{-1} = A_{d-n}$. Since $\text{sign}(\det(A)) C_n$ is conjugate (over \mathbb{Z}) to B_n , it follows that $\text{sign}(\det(A)) A_{d-n}$ is conjugate (over \mathbb{Z}) to B_n .

Now Theorem 5.5 can be restated, in terms of the matrices B_n 's, as in the following proposition. This is exactly Theorem 4.9 of [EaHR10].

Theorem 5.5 *Let $A \in M_d(\mathbb{Z})$ be an integer dilation matrix. The K -groups of the Cuntz-Li algebra \mathcal{U}_{A^t} are as follows.*

1. *If d is even and $\det(A) > 0$, then*

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \text{coker}(1 - B_n), \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \text{coker}(1 - B_n) \oplus \mathbb{Z}.$$

2. If d is even and $\det(A) < 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n), \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n).$$

3. If d is odd and $\det(A) > 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n) \oplus \mathbb{Z}, \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n).$$

4. If d is odd and $\det(A) < 0$ then

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n), \text{ and}$$

$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \leq n \leq d}} \operatorname{coker}(1 - B_n).$$

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